Firm Size Distribution and Power Laws in Brazil: Some Empirical Evidence

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Abstract

The paper investigates power laws for firm size distribution in Brazil. It does not assume the initial validity of a simpler power law pattern. A stronger support emerges for the extreme upper tail of the distributions relatively to a Pareto type II model. The patterns are not uniform and prevail for at most 86% of the years for Pareto type I embedded in a Pareto type II model and the consistency with the Zipf law would be observed for 76% of the years. It is possible that more complex models are not necessarily driven by a single shape parameter.

Keywords: power law; firm size; Pareto law; Brazil

JEL Classification: C12; C49; L11

1. Introduction

The search for distributional regularities is valuable as an initial characterization of phenomena in different areas of knowledge. An influential example in economics relates to lognormal limiting distributions for firm size associated with the so-called Gibrat´s law as implied by rates of firm growth for firms that are size-independent (see Sutton 1997 and Coad 2009). Quite often, regularities take the form of a power law that can be expressed as $Y = \alpha X^{\beta}$ with Y and X denoting variables of interest and α standing for a typically uninteresting constant, whereas the shape parameter β takes the center of the analysis. The relation is ascribed for the upper tail of the distribution, and it is remarkable that such regularities appear to prevail in different contexts as in city sizes, firm sizes and stock market movements (see Gabaix 2009; 2016 for overviews). The major challenge is to move further and uncover the underlying generating mechanisms that may eventually be attributed to well delineated economic processes.

The present paper focus on different aspects of the firm size distribution in Brazil. A handful of studies on Gibrat´s law or its distributional implications were undertaken by Resende (2005), Esteves (2007), Ribeiro (2007) and Resende and Cardoso (2013). The evidence strongly rejects the prevalence of Gibrat´s law, except for some favorable results for the upper tail of the firm size distribution in one study. However, there is still potential for additional investigations on topics related to the firm size distribution that include power law features. In any case, dynamic aspects involving distributional changes and possible explanatory factors are likely to be relevant, especially in complex and heterogeneous economies. In fact, the relevance of possible long-run distributional shifts is discussed by Cabral and Mata (2003) for the case of a smaller economy like Portugal. The evolution of the firm size distribution evolves towards a lognormal pattern and the calibration of a theoretical model, that incorporates financial constraints, displayed empirical consistency. Angelini and Generale (2008) further explored the role of financial constraints for the firm size distribution in Italy. Lotti and Santarelli (2004) also investigate firm size distributions in the Italian manufacturing industry. Newly created firms were followed on a quarterly base for six years and nonparametric methods were applied to different sectors. Altogether, the evidence that emerges from the aforementioned studies mostly indicates right skewed firm size distributions. Even though, one observes different convergence patterns of the distributions over time in the latter study while in principle some consistency with the mechanisms addressed in the literature on firm selection and industry evolution appears to prevail. Luttmer (2007) considered the underlying firm-specific heterogeneities and the related implementation with calibration methods favored a limiting firm size distribution that is a particular case of a Pareto distribution, namely the Zipf distribution.[1]

As for dynamic power law aspects it is worth mentioning Fernholz (2017), that proposes nonparametric econometric methods that characterize general power law distributions under basic stability conditions. Although the necessary requirements are not always met in all settings, it indicates that analyses that consider potentially changing distributions that are prone to a power law pattern may be fruitfully conducted.

Even when power law patterns are detected, the next natural step involves the discussion of possible generating mechanisms. Kumamoto and Kamihigashi (2018) attempt to map the possible classes of stochastic processes that can be consistent with power law patterns. Gabaix (2009, 2016) highlights underlying mechanisms associated with random growth, performance aspects related to extreme upper tails of the distribution ("economics of superstars") and transfer of power laws through aggregation.[2]

The study of power laws for firm size distribution still warrants further assessments. In the case of Brazil, Da Silva et al. (2018) addressed the prevalence of power laws for firm size, as measured by net revenue shares, in the case of the largest 1000 firms in Brazil in 2015. For that purpose, Pareto and Gumbel exponents were obtained upon the estimation of these

[1] It was initially conceived by Zipf (1949). An overview can be found in Saichev et al. (2010).
[2] Gabaix (1999) provides an attempt of uncovering generating mechanisms for the Zipf law in the case of city sizes.

assumed laws and the evidence is claimed to properly fit a Pareto law that is roughly consistent with the particular case of a Zipf law.[3] However, the upfront assumption of a simple power law process is questionable, although one should be less concerned on biases relating to OLS estimations given the large sample size in that particular application.[4] Furthermore, the authors emphasize the role of large firms in the Brazilian economy by which microeconomic shocks could have significant macroeconomic implications.[5] In fact as power laws relate to the upper portions of the size distribution, the so-called granular aspect may have important policy implications (see Rossi-Hansberg and Wright 2007 and Gabaix 2011).

Finally, it is conceivable that distributional characteristics for firm size may change over a longer time horizon. For example, Rodrik (2016) contends that developing economies could be subject to premature deindustrialization. A related aspect may be associated with the reduction of the diversification of the economy. Hutchinson et al. (2010) find that in the UK and Belgium 4-digit industries, inter-industry diversification leads to a shift of the firm size distribution to the right with a stronger effect observed in the case of older age groups. Conversely, the possibility of a less diversified industry in Brazil, for example, could hypothetically lead to changes in power law patterns, if those are relevant. In fact, the relative loss of importance of specific sectors can potentially not only change firm size distributions but possibly the shape parameter of a power law, which typically holds a negative relation with firm size inequality. Such conjecture needs to be empirically addressed.

The present paper aims at contributing to the literature in at least two aspects:

a) By not assuming a simpler distribution, say a Pareto type I or yet a Zipf, from the start, but rather nest those models within a more general model, Thus, one is able to test the

[3] In many empirical applications, the OLS-based procedure could lead to significant biases in small samples as indicated by Gabaix and Ibragimov (2011), who suggested a possible correction. Furthermore, Urzúa (2011) had stressed, in the context of the particular case of the Pareto's law given by the Zipf´s law, that the referred procedure would be inappropriate, as the intercept is not a nuisance parameter in the regression. Schluter (2021) further elaborates on the estimation bias arising from OLS-based estimations for the Zipf law.

[4] Coad (2010) emphasizes the importance of not assuming more restrictive hypotheses when testing for the Pareto law.

[5] The oil firm Petrobrás is by far the largest in Brazil and operates in an extremely volatile sector.

restricted model in a maximum likelihood framework and assess whether it is meaningful that a single shape parameter can drive the firm size distribution in Brazil;

b) By considering a long time period (1999-2019) that, in principle, could be subject to important structural changes

The remainder of the paper is organized as follows. The second section discusses basic conceptual aspects related to power law testing. The third section discusses the data and the obtained empirical results for the particular empirical application. The fourth section concludes.

2. Power laws and firm size distributions: empirical aspects

2.1- Basic conceptual aspects

An influential power law emerges in the literature pertaining to the Pareto distribution that can be traced back to Pareto (1897) in the context of the assessment of income distribution. Useful overviews can be found in Johnson and Kotz (1970, Chap. 19). Specifically, the so-called Pareto Type I distribution relates to a strictly positive continuous random variable $x \in \Re_{++}$ with probability density function (pdf) given by: [6]

$$f_p(x) = \frac{\theta}{\mu}\left[\frac{\mu}{x}\right]^{\theta+1} \qquad x \geq \mu > 0, \theta > 0 \qquad (1)$$

where μ stands for the location parameter and corresponds to the minimum value of x that indicates where the so-called “heavy tail" starts, whereas θ indicates the shape parameter that constitutes the main parameter of interest and is often known as the Pareto exponent. A salient feature relates to dispersion measures holding an inverse relation with the shape parameter θ. In particular, the coefficient of variation is given by $CV(X) = 1/\sqrt{\theta(\theta-2)}$ for $\theta > 2$, where the sole dependence on the shape parameter reflects a relative character.

[6] Throughout the paper, we try to make the different notations used in the literature uniform. In particular, the shape parameter is denoted by θ instead of α.

A fruitful research strategy for testing power laws is to consider more general probability distributions that nest the Pareto type I. Goerlich (2013) suggests the consideration of a more general case given by a member of the Burr (1942) family of distributions [see Johnson and Kotz 1970, Chap. 12.4.5 for an overview], also known as a Pareto type II distribution (see Arnold 2015). That more general distribution is characterized by the following probability density function (pdf) given by:

$$f_B = \frac{\theta}{\sigma}(1 + \frac{x-\mu}{\sigma})^{-(\theta+1)} \qquad x \geq \mu > 0, \theta > 0, \sigma > 0 \qquad (2)$$

Note that such function nests the case of a Pareto type I distribution when $\sigma = \mu$ or even, more strongly, would nest the particular case of the Zipf law where $\sigma = \mu$ and $\theta = 1$. Moreover, it is worth noting that in contrast with the Pareto type I case, expression. (2) indicates the presence of an additional parameter given by σ . In that case, the dispersion measure of the coefficient of variation is given by $\frac{\theta}{(1-\theta)^2(\theta-2)}\sigma^2$ for $\theta > 2$ and therefore σ exerts a positive influence on dispersion despite the negative effect accruing from θ.[7]

Goerlich (2013) proposes to test the null hypothesis of H_0: $\sigma = \mu$ in (2) by means of a Lagrange multiplier (LM) test that requires only the maximum likelihood estimation of the restricted model, the Pareto type I case. Thus proceeding, one can obtain the following expression:

$$LM_P = n\frac{(\tilde{\theta}+2)(\tilde{\theta}+1)^4}{\tilde{\theta}}\tilde{z}^2 \sim_{asy} \chi^2(1) \; under \; H_0\colon \sigma = \mu \qquad (3)$$

where $\tilde{z} = \frac{\tilde{\theta}}{\tilde{\theta}+1} - \frac{1}{n}\sum_{i=1}^{n}\frac{\mu}{x_i}$ with $\tilde{\sigma} = \mu$ and $\tilde{\theta}=\hat{\theta}$ denoting restricted esti1mates. The computation of the test σταιστιχ is simple given the maximum likelihood estimates of $\hat{\theta}$ obtained for the Pareto type I model. The author presents evidence that indicates satisfactory results from Monte Carlo simulations with good small sample properties and adequate power against some plausible alternatives.

[7] The reader is referred to the aforementioned overviews for further details on the moments and different characteristics of the Burr (also known as Pareto type II) and Pareto type I distributions.

Previously, Urzúa (2000) had advanced an analogous testing strategy for the Zipf law that considers the more specific null hypothesis of H_0: $\sigma = \mu,\ \theta = 1$. Accordingly, it is possible to obtain the Lagrange multiplier (LM) test statistic that is given as follows:

$$LM_Z = 4n[z_1^2 + 6z_1 z_2 + 12z_2^2] \sim_{asy} \chi^2(2) \quad under\ H_0\!: \sigma = \mu\,, \theta = 1 \qquad (4)$$

where $z_1 \equiv 1 - \frac{1}{n}\sum_{i=1}^{n} ln(\frac{x_i}{\mu})$ and $z_2 \equiv \frac{1}{2} - \frac{1}{n}\sum_{i=1}^{n} \frac{\mu}{x_i}$

Note that now one has an additional restriction under the null hypothesis in contrast with (3) and therefore a distinct number of degrees of freedom in the asymptotic distribution. Furthermore, the reported finite sample properties for the LM_Z test as obtained from Monte Carlo simulations appear as adequate.

Thus, the nested testing approaches that were just discussed suggest a sequential testing strategy that first obtains LM_P and then, if the evidence does not favor the more general Pareto type II distribution, one would obtain LM_Z.

2.2 – More general nesting structures

Urzúa (2020) argues that the contrasts between the null and alternative hypotheses would be subtle under the testing strategies advanced by Urzúa (2011) and Goerlich (2013) as the involved distributions would have a similar heavy-tail behavior. Thus, he suggests embedding the Pareto model within a more general structure of a Pareto type IV model. The more general Pareto type IV density can be expressed as follows [see Arnold (2015)]:

$$f(x) = \frac{\theta}{\gamma\sigma}[1 + (\frac{x-\mu}{\sigma})^{\frac{1}{g}}]^{-(\theta+1)}(\frac{x-\mu}{\sigma})^{\frac{1}{\gamma}-1} \qquad for\ \ x > \mu \qquad (5)$$

One can note an additional inequality parameter $\gamma > 0$. The imposition of the restrictions $\sigma = \mu$ and $g = 1$ would readily lead to the Pareto type I model as given by expression (1). Additionally, if $\theta = 1$, one would have the stricter case consistent with a Zipf law.

The author undertakes an analogous testing procedure as indicated by the aforementioned works but with the more general alternative provided by a Pareto IV model. It can be shown that the corresponding Lagrange multiplier test is given as follows:

$$PWL = \hat{d}'\hat{I}^{-1}\hat{d} \sim_{asy} \chi^2(2) \quad under\ H_0: \sigma = \mu, \gamma = 1 \qquad (6)$$

Where $\hat{d}$ = (d_1,d_2,0)´ with

$$d_1 = \frac{n\hat{\theta}}{\mu} - (\hat{\theta}+1)\sum_{i=1}^{n}\frac{1}{x_i} \qquad (7)$$

$$d_2 = -n + \hat{\theta}\sum_{i=1}^{n} ln\left(\frac{x_i-\mu}{\mu}\right) - (\hat{\theta}+1)\sum_{i=1}^{n}\frac{\mu}{x_i} ln\left(\frac{x_i-\mu}{\mu}\right) \qquad (8)$$

and yet with $\hat{I}^{-1}$ denoting the inverse of the information matrix.[8]

Later in the empirical analysis in section 3.2, we will implement the PWL test by means of the *pwlaw* Stata module developed by Urzúa (2020).

3. Empirical Analysis

3.1 - Data

This study makes use of balance sheet data for the 1000 largest firms in Brazil in an annual basis for the period 1999-2019. The coverage of the considered data sources is broader than more usual databases that focus only on publicly traded firms. In fact, the pioneering databases were constructed in connection with the periodical Conjuntura Econômica [Instituto Brasileiro de Economia, Fundação Getulio Vargas – IBRE-FGV] in terms of an annual special issue for the largest 500 firms and later a database for the 1000 largest firms was established while similar efforts were initiated by newspaper Valor Econômico. We were able to obtain data for 1999 and 2000 from IBRE-FGV and for 2001-2019 from Valor Econômico for the 1000 largest firms.[9]

In the present study, the variable of interest will be the net revenue share of firm i in year (rs_{it}) computed upon the share of net revenue (NR) relative to the year's total value, which is readily defined for the n largest firms:

$$rs_{it} = \frac{NR_{it}}{\sum_{i=1}^{n} NR_{it}}$$

[8] Urzúa (2020) builds on Brazauskas (2002) fo obtaing that component for the test statistic. The reader is refereed to that work as the matrix is omitted for conciseness in the present paper.
[9] The used data can be provided upon request.

The power law analyses will be implemented on a year-by-year basis for the referred upper tail of the distribution of net revenue shares and not always the top firms are the same in different years.

3.2 - Empirical results

The empirical strategy considers the test of a Pareto type I distribution that is nested by some more general distribution. Table 1 presents evidence from the previously mentioned Lagrange multiplier tests proposed by Goerlich (2013) and Urzúa (2000). The referred score tests only consider restricted maximum likelihood estimates of a Pareto type I model. The results were obtained with the Stata module PARETOFIT [See Jenkins and Van Kerm (2015)].

INSERT TABLE 1 AROUND HERE

The evidence for the largest 1000 firms in Brazil clearly favors the rejection of a Pareto type I as contrasted with a Pareto type II as indicated by the LM_P test statistics in all years along 1999-2019. The LM_Z test considers a stricter version of the Pareto type I model with the null hypothesis of $\theta = 1$ (the Zipf distribution). Given the stronger assumption, it is clear that the evidence should favor the rejection of the restricted model. Nevertheless, the results for LM_Z are also reported for completeness.

Thus, for the largest 1000 firms in Brazil, the evidence is not consistent with a simple power law pattern. However, power laws typically prevail for the upper tail of some distributions of variables of interest. A further investigation possibility could consider a higher cutoff point for the Pareto distribution. In fact, in the context of firm size, another possibility is to focus on the 500 largest firms. The corresponding results for the tests are reported in Table 2.

INSERT TABLE 2 AROUND HERE

Under this stricter criterion, the evidence is partially distinguishable. The evidence for rejection of a Pareto type I law is indicated for most pf the years [1999-2012 and 2017-

2019]. However, one cannot reject the null hypothesis for the 2013-2016 period as suggested by the LM_P test. Additionally, evidence from the LM_Z test favors the prevalence of a Zipf law for that upper tail of the firm size distribution in Brazil for the 2013-2016 years.

Finally, in Table 3, we consider a higher cutoff in terms of the 100 largest firms in Brazil. In such case, a simple power law pattern, with a Pareto type I law, emerges in most years [1999-2016]. In fact, in those years the evidence is consistent with a Pareto type I and also with the particular case of Zipf law (with $\theta = 1$) while during the 2017-2019 period, the evidence favors a Pareto type II distribution that does not display a simple power law pattern driven by a single shape parameter.

INSERT TABLE 3 AROUND HERE

Table 4 reports the estimates for the shape parameter θ. Despite the relatively long period, one does not observe significant changes over time. In 86% of the studied years there is support for Pareto type I model embedded in a Pareto type II model, whereas the consistency with stricter case of the Zipf law would be observed for 76% of the years, albeit with not entirely consistent patterns in relation to the two different models in a few years.

INSERT TABLE 4 AROUND HERE

Furthermore, the relatively stable trajectory of θ with values close or statistically equal to unity does not conform with a previous conjecture that deindustrialization trends could potentially lead to significant changes in the Pareto exponent.

Finally, as discussed in section 2.2, it may be pertinent to nest the Pareto type I model within the more general model of a Pareto type IV alternative. In Table 5, we present the results for the aforementioned test PWL. In that case, favorable evidence for a power law pattern is much weaker and only prevails for 29% of the investigated years [2000, 2003, 2005, 2013-15].

INSERT TABLE 5 AROUND HERE

4. Final comments

The paper investigated distributional properties of the largest firms in Brazil during 1999-2019, taking as a reference net revenue shares. The main focus of the analysis refers to the assessment of the prevalence of simple power patterns where a single shape parameter properly portrays the firm size distribution for its upper tail [1000, 500 and 100 largest firms]. Specifically, the Pareto type I case or its particular case of the Zipf law were nested either within a Pareto type II or a yet a more general Pareto type IV model. The evidence, thus obtained for a power law, is only strong for the upper tail of the distribution corresponding to the largest 100 firms. In that portion, the case for a Pareto type model and even a Zipf model is appealing for a significant portion of the studied years. However, the results are weaker under a less strict cutoff point and for a more general alternative distribution.

Altogether, the prevalence of a simple power law pattern is not completely clear-cut in the case of firm size distribution in Brazil. Therefore, a potentially fruitful avenue for future research could involve the maximum likelihood of more general models; say the Pareto IV model and a detailed statistical assessment for different years. Furthermore, should reliable data become available since the 80s, comparative distributional analyses could be undertaken for potentially capturing structural changes that might have taken place during a longer time interval.

Finally, either if a clear power law emerges or not, a better understanding of firm-level growth can be relevant. In the case of countries with especially dominant firms, the features of the firm size distribution are relevant for understanding the propagation of microeconomic shocks and its macroeconomic effects. Thus, irrespective of the prevalence of power law patterns, different features associated with the firm size distribution in Brazil warrant further investigations.

5. Declaration

5.1 - Funding and/or conflicts of interests/competing interests

The authors declare that the current submitted work did not have any funding and that there are no conflicts of interest of any kind in the elaboration of such work

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Table 1

Lagrange multiplier tests (LM) for restricted Pareto Type II distributions for the 1000 largest firms in Brazil – 1999-2019

Year	LM_P [H_0: $\sigma = \mu$]		LM_Z [H_0: $\sigma = \mu$, $\theta = 1$]	
	test statistic	p-value	test statistic	p-value
1999	47.445	0.000	181762	0.000
2000	27.870	0.000	54.895	0.000
2001	31.541	0.000	51.825	0.000
2002	24.685	0.000	35.554	0.000
2003	34.206	0.000	55.244	0.000
2004	23.200	0.000	44.346	0.000
2005	15.632	0.000	33.384	0.000
2006	8.084	0.004	21.541	0.000
2007	10.534	0.001	20.956	0.000
2008	12.745	0.000	27.074	0.000
2009	7.165	0.007	16.866	0.000
2010	10.305	0.001	20.231	0.000
2011	20.807	0.000	67.855	0.000
2012	15.194	0.000	59.068	0.000
2013	23.962	0.000	71.820	0.000
2014	29.130	0.000	79.676	0.000
2015	56.263	0.000	127.042	0.000
2016	31.692	0.000	88.075	0.000
2017	20.896	0.000	83.457	0.000
2018	54.399	0.000	151.423	0.000
2019	30.702	0.000	111.622	0.000

Table 2

Lagrange multiplier tests (LM) for restricted Pareto Type II distributions for the 500 largest firms in Brazil – 1999-2019

Year	LM_P [H_0: σ = μ]		LM_Z [H_0: σ = μ , θ = 1]	
	test statistic	p-value	test statistic	p-value
1999	24.662	0.000	32.017	0.000
2000	8.045	0.002	8.036	0.018
2001	10.619	0.001	11.082	0.004
2002	10.957	0.001	10.968	0.004
2003	12.934	0.000	12.759	0.002
2004	12.134	0.000	12.712	0.002
2005	6.668	0.010	6.810	0.033
2006	8.111	0.004	8.305	0.016
2007	10.090	0.002	10.887	0.004
2008	9.602	0.002	10.879	0.004
2009	8.467	0.004	9.564	0.008
2010	8.227	0.004	8.271	0.016
2011	6.950	0.008	12.185	0.002
2012	3.892	0.048	7.978	0.019
2013	3.704	0.054	5.292	0.071
2014	2.214	0.137	2.866	0.239
2015	2.683	0.101	3.013	0.222
2016	3.032	0.082	4.503	0.105
2017	9.741	0.002	14.271	0.001
2018	8.703	0.003	12.725	0.002
2019	5.332	0.021	9.1892341	0.010

Table 3

Lagrange multiplier tests (LM) for restricted Pareto Type II distributions for the 100 largest firms in Brazil – 1999-2019

Year	**LM_P [H_0: $\sigma = \mu$]**		LM_Z [H_0: $\sigma = \mu$, $\theta = 1$]	
	test statistic	**p-value**	**test statistic**	**p-value**
1999	3.395	0.065	5.574	0.062
2000	0.247	0.619	5.088	0.079
2001	0.254	0.615	7.079	0.029
2002	1.157	0.282	5.051	0.080
2003	1.086	0.297	4.704	0.095
2004	1.424	0.233	4.541	0.103
2005	2.094	0.148	4.679	0.096
2006	1.362	0.243	4.018	0.134
2007	1.274	0.259	4.887	0.087
2008	2.521	0.112	4.246	0.120
2009	0.534	0.465	4.437	0.109
2010	1.345	0.246	4.103	0.129
2011	2.158	0.142	2.185	0.335
2012	0.456	0.500	1.252	0.535
2013	0.797	0.372	1.433	0.488
2014	1.299	0.254	1.774	0.412
2015	1.807	0.179	2.800	0.247
2016	0.598	0.439	1.460	0.482
2017	5.229	0.022	5.027	0.081
2018	5.290	0.022	6.505	0.038
2019	8.725	0.003	8.825	0.012

Table 4

Pareto exponents for the 100 largest companies in Brazil – 1999-2019

Year	Shape parameter (θ)	p-value
1999	1.261*	0.000
2000	1.291*	0.000
2001	1.359**	0.000
2002	1.283**	0.000
2003	1.270*	0.000
2004	1.257*	0.000
2005	1.248*	0.000
2006	1.234*	0.000
2007	1.275*	0.000
2008	1.212*	0.000
2009	1.266*	0.000
2010	1.238*	0.000
2011	1.049*	0.000
2012	1.109*	0.000
2013	1.102*	0.000
2014	1.100*	0.000
2015	1.148*	0.000
2016	1.116*	0.000
2017	1.053****	0.000
2018	1.043***	0.000
2019	0.992***	0.000

Notes: following resilts from Table 3:

(*) consistent with both Pareto type I and with Zipf laws

(**) consistent with Pareto type I but not with Zipf law

(***) not consistent even with Pareto type I and therefore θ is not the only relevant parameter [see Table 3]

(****) ambigous case with only the acceptance of Zipf law but with borderline p-values

Table 5

Lagrange multiplier tests (LM) for restricted Pareto Type IV distributions for the 100 largest companies in Brazil – 1999-2019

Year	PWL [H_0: $\sigma = \mu$, $\gamma = 1$]	
	test statistic	´p-value
1999	10.603	0.005
2000	3.659	0.160
2001	46.495	0.000
2002	178.021	0.000
2003	9.122	0.104
2004	32.622	0.000
2005	9.180	0.102
2006	47.133	0.000
2007	9.178	0.010
2008	17.956	0.000
2009	28.121	0.000
2010	7.295	0.026
2011	7.273	0.026
2012	144.482	0.000
2013	4.952	0.084
2014	4.307	0.116
2015	5.880	0.053
2016	6.477	0.039
2017	9.237	0.010
2018	9.962	0.007
2019	13.130	0.001